Tutorial Notes 10

1. Find a parametrization of the spherical cap $x^2 + y^2 + z^2 = 8$, $z \geq -2$.

   Solutions:

   We use spherical coordinates. Then a parametrization is
   $$(2\sqrt{2}\sin\phi\cos\theta, 2\sqrt{2}\sin\phi\sin\theta, 2\sqrt{2}\cos\phi), \quad 0 \leq \phi \leq \frac{3\pi}{4}, 0 \leq \theta \leq 2\pi.$$

2. Find the area of the portion of the plane $y + 2z = 2$ inside the cylinder $x^2 + y^2 = 1$.

   Solutions:

   Let $F(x, y, z) = y + 2z$, then the surface is on the level set $F(x, y, z) = 2$ where $x^2 + y^2 \leq 1$. Then
   $$\mathrm{d}S = \frac{\sqrt{5}}{2}\,\mathrm{d}x\,\mathrm{d}y.$$
   Hence the area is
   $$\int_{x^2+y^2\leq 1} \frac{\sqrt{5}}{2}\,\mathrm{d}x\,\mathrm{d}y = \frac{\sqrt{5}}{2}\pi.$$

3. (a) Find a generalized spherical coordinate parametrization of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

   (b) Write the integral of the surface area.

   Solutions:

   (a) A spherical coordinate parametrization is
   $$(a\sin\phi\cos\theta, b\sin\phi\sin\theta, c\cos\phi), \quad 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi.$$

   (b) Using the above parametrization,
   $$r_\phi = (a\cos\phi\cos\theta, b\cos\phi\sin\theta, -c\sin\phi)$$
   $$r_\theta = (-a\sin\phi\sin\theta, b\sin\phi\cos\theta, 0).$$
   Then
   $$r_\phi \times r_\theta = (-bc\sin^2\phi\cos\theta, ac\sin^2\phi\sin\theta, ab\sin\phi\cos\phi)$$
   and
   $$\mathrm{d}S = \sqrt{c^2\sin^4\phi(b^2\cos^2\theta + a^2\sin^2\theta) + a^2b^2\sin^2\phi\cos^2\phi}\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

The surface area is

$$\int_0^{2\pi}\int_0^{\pi}\sqrt{c^2\sin^2\phi(b^2\cos^2\theta+a^2\sin^2\theta)+a^2b^2\cos^2\phi}\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

4. Evaluate

$$\int_S x^2\sqrt{5-4z}\,\mathrm{d}S,$$

where S: $z=1-x^2-y^2$, $z\geq 0$.

Solutions:

$$\mathrm{d}S=\sqrt{1+4x^2+4y^2}\,\mathrm{d}x\,\mathrm{d}y.$$

Then

$$\begin{aligned}\int_S x^2\sqrt{5-4z}\,\mathrm{d}S &= \int_{x^2+y^2\leq 1} x^2(\sqrt{1+4x^2+4y^2})^2\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{x^2+y^2\leq 1} x^2(1+4x^2+4y^2)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{x^2+y^2\leq 1} \frac{x^2+y^2}{2}(1+4x^2+4y^2)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^{2\pi}\int_0^1 \frac{r^2(1+4r^2)}{2}r\,\mathrm{d}r\,\mathrm{d}\theta\\ &= \frac{11\pi}{12}.\end{aligned}$$

5. Find the flux of the vector field $(y^2, xz, -1)$ outward through the cone $z=\sqrt{x^2+y^2}$, $0\leq z\leq 2$.

Solutions:

Since the cone is a portion of $x^2+y^2-z^2=0$, an outward normal vector is

$$(2x, 2y, -2z),$$

whose unit vector is

$$\frac{(x,y,-z)}{\sqrt{2}z}.$$

Moreover,

$$\mathrm{d}S=\sqrt{1+z_x^2+z_y^2}\,\mathrm{d}x\,\mathrm{d}y=\sqrt{2}\,\mathrm{d}x\,\mathrm{d}y.$$

Hence the flux is

$$\begin{aligned}&\int_{x^2+y^2\leq 4}(y^2,xz,-1)\cdot\frac{(x,y,-z)}{\sqrt{2}z}\sqrt{2}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{x^2+y^2\leq 4}\frac{xy^2+xyz+z}{z}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{x^2+y^2\leq 4}\mathrm{d}x\,\mathrm{d}y\\ &= 4\pi.\end{aligned}$$